Math 72 9.4 Properties of Logarithms and the change-of base formula

Math 62 10.4

More Properties of Logarithms

Objectives

"combine"-type problems

1) Combine $\log_b a + \log_b c$ using property
2) Combine $\log_b a - \log_b c$ using property
3) Repeatedly combine $\log_b a + \log_b a + \ldots = k\log_b a$
4) Use these properties together to combine to a single log with coefficient 1.

"separate"-type problems.

5) Write a single log as a sum, difference and/or multiple of simpler logs using the same properties, in the opposite direction.
6) Given numerical values for a few simple logs, separate a more complicated log into those simpler logs to find its numerical value.

7) Use the change-of-base formula
   - to calculate $\log_b(a)$ where $b \neq 10$ or e
   - to graph $y = \log_b(x)$ where $b \neq 10$ or e

(More) Properties of Logarithms.

Consider the sum of two logs, same base, b:

$$\log_b x + \log_b y$$

Let's make a substitution to make this simpler.

$M = \log_b x \longleftrightarrow b^M = x$

$N = \log_b y \longleftrightarrow b^N = y$

$$= M + N$$

Using the inverse property:

$$= \log_b b^{M+N}$$

Rules of exponents tell us:

$$= \log_b (b^M \cdot b^N)$$

Substitute back:

$$= \log_b (x \cdot y)$$

Beginning = End:

$$\boxed{\log_b x + \log_b y = \log_b xy}$$

← Sometimes called the Product property of logs.

Similarly:

$$\log_b x - \log_b y$$

$$= M - N$$

$$= \log_b b^{M-N}$$

$$= \log_b \left(\frac{b^M}{b^N}\right)$$

$$= \log_b \left(\frac{x}{y}\right).$$

Giving

$$\boxed{\log_b x - \log_b y = \log_b \left(\frac{x}{y}\right)}$$

← Sometimes called the Quotient Property of Logs.

Remember $x + x + x = 3x$ multiplication is repeated addition.

So $\log_b x + \log_b x + \log_b x = 3 \log_b x$.

Using the same methods as before:

$$
\begin{aligned}
& 3 \log_b x \\
&= 3M && \text{Subst} \\
&= \log_b b^{3M} && \text{inverse property.} \\
&= \log_b((b^M)^3) && \text{exponent rules.} \\
&= \log_b(x^3). && \text{subst back}
\end{aligned}
$$

$\log_b x = M$ means $b^M = x$.

This means

$$3 \log_b x = \log_b x^3$$

We can expand this for any multiple k:

$$\boxed{k \cdot \log_b x = \log_b x^k}$$

This is sometimes called the Power Rule for exponents.

We can see these properties if we put numbers/expressions in

$$\log(100) + \log\left(\tfrac{1}{10}\right) \stackrel{?}{=} \log\left(100 \cdot \tfrac{1}{10}\right)$$

$$\downarrow$$

$$2 \quad + \quad (-1) \qquad \log(10)$$

$$2 - 1 = 1 \checkmark$$

$$\log(1000) - \log(10) \stackrel{?}{=} \log\left(\tfrac{1000}{10}\right)$$

$$3 \quad - \quad 1 = \log(100)$$

$$3 - 1 = 2 \checkmark$$

$$\log(10^5) = 5 \cdot \log 10$$

$$5 = 5 \cdot 1 \checkmark$$

OR with base e....

$$\ln(e^3) + \ln(e^4) \stackrel{?}{=} \ln(e^3 \cdot e^4)$$

$$3 + 4 = \ln(e^{3+4})$$ exp law → add exp

$$= \ln e^7$$

$$3 + 4 = 7 \checkmark$$

$$\ln(e^6) - \ln(e^2) \stackrel{?}{=} \ln\left(\frac{e^6}{e^2}\right)$$ exp law – subtract exp

$$6 - 2 = \ln(e^{6-2})$$

$$= \ln(e^4)$$

$$6 - 2 = 4 \checkmark$$

$$\ln(e^6) \stackrel{?}{=} 6 \cdot \ln e$$

$$6 = 6 \cdot 1 \checkmark$$

Write each expression as a single log with coefficient 1.

① $\log_3\left(\frac{1}{2}\right) + \log_3(12)$

$= \log_3\left(\frac{1}{2}\cdot 12\right)$

$= \boxed{\log_3(6)}$ or $\boxed{\log_3 6}$

② $\log 27 - \log 3$

$= \log\left(\frac{27}{3}\right)$

$= \boxed{\log 9}$ ← no base? It's base 10, common log.

③ $2\log_7 3$

$= \log_7 3^2$

$= \boxed{\log_7 9}$

④ $\log_2(x+2) + \log_2 x$

$= \log_2\left[(x+2)\cdot x\right]$

$= \boxed{\log_2(x^2+2x)}$

note: ( ) are required.
$\log_2 x^2 + 2x$ means $\log_2(x^2) + 2x$
or $2x + \log_2(x^2)$.

⑤ $\log_3(x^2+5) - \log_3(x^2+1)$

$= \boxed{\log_3 \frac{x^2+5}{x^2+1}}$

⑥ $4\log_3 x$

$= \boxed{\log_3 x^4}$

$2\log_5 3 + 3\log_5 2$

$= \log_5 3^2 + \log_5 2^3$

$= \log_5 9 + \log_5 8$

$= \log_5 9 \cdot 8$

$= \boxed{\log_5 72}$

(8) $3\log_9 x - \frac{1}{2}\log_2(x+1)$

$= \log_9 x^3 - \log_2 (x+1)^{\frac{1}{2}} = \boxed{\log_9 x^3 - \log_2 \sqrt{x+1}}$

different bases! cannot combine further.

(9) $\log_4 25 + \log_4 3 - \log_4 5$

$= \log_4 (25 \cdot 3) - \log_4 5$

$= \log_4 75 - \log_4 5$

$= \log_4 \frac{75}{5}$

$= \boxed{\log_4 15}$

(10) $5\log_6 x - \frac{3}{4}\log_6 x + 3\log_6 x$

Quick kill: combine like terms!

$= (5 - \frac{3}{4} + 3)\log_6 x$

$= \frac{29}{4}\log_6 x$

$= \boxed{\log_6 x^{29/4}}$

Not such a quick kill –

$$= \log_6 x^5 - \log_6 x^{3/4} + \log_6 x^3$$

$$= \log_6 \frac{x^5}{x^{3/4}} + \log_6 x^3$$ Subtract exp $5 - \frac{3}{4} = \frac{17}{4}$

$$= \log_6 x^{17/4} + \log_6 x^3$$

$$= \log_6 x^{17/4} \cdot x^3$$ add exp $\frac{17}{4} + 3$

$$= \boxed{\log_6 x^{29/4}}$$

(11) $2\log_5 x + \frac{1}{3}\log_5 x - 3\log_5 (x+5)$

$= \log_5 x^2 + \log_5 x^{1/3} - \log_5 (x+5)^3$

$= \log_5 \dfrac{x^2 \cdot x^{1/3}}{(x+5)^3}$

add exponents

$2 + \frac{1}{3}$

$= \frac{6}{3} + \frac{1}{3}$

$= \frac{7}{3}$

$= \boxed{\log_5 \dfrac{x^{7/3}}{(x+5)^3}}$

(12) $2\log_7 y + 6\log_7 z$

$= \log_7 y^2 + \log_7 z^6$

$= \boxed{\log_7 (y^2 z^6)}$

SUMMARY OF LOG PROPERTIES $\quad b \neq 1,\ b > 0$

1) $\log_b 1 = 0$

2) $\log_b b^x = x$

3) $b^{\log_b x} = x$

4) $\log_b x \cdot y = \log_b x + \log_b y$

5) $\log_b \dfrac{x}{y} = \log_b x - \log_b y$

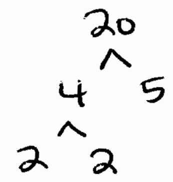
6) $\log_b x^k = k \cdot \log_b x$

"Separate"-type problems

Write each log as a sum of logs.

① $\log_3(20)$

<u>step 1</u>: Factor 20 to prime factors



$20 = 2^2 \cdot 5$

$$\log_3(20) = \log_3(2^2 \cdot 5)$$

<u>step 2</u>: use properties 1 & 2 first

$$= \log_3(2^2) + \log_3(5)$$

<u>step 3</u>: use property 3 last

$$= \boxed{2\log_3(2) + \log_3(5)}$$

or $\boxed{2\log_3 2 + \log_3 5}$

② $\log_2(4y) = \log_2(2^2 \cdot y)$

$= \log_2(2^2) + \log_2 y$ ← remember inverse functions! $\log_2(2^x) = x$

$= \boxed{2 + \log_2 y}$

③ $\log_5(ab) = \boxed{\log_5 a + \log_5 b}$

Write each log as a difference of logs

(4) $\log_3\left(\frac{5}{4}\right) = \log_3(5) - \log_3(4)$

$= \log_3(5) - \log_3(2^2)$

$= \boxed{\log_3 5 - 2\log_3 2}$

(5) $\log_2\left(\frac{8}{y}\right) = \log_2(8) - \log_2(y)$

$= \log_2(2^3) - \log_2(y)$

$= \boxed{3 - \log_2 y}$

inverses again!

$\log_2 2^x = x$

(6) $\log_5\left(\frac{a}{b}\right) = \boxed{\log_5 a - \log_5 b}$

Write each log as the multiple of a log

(7) $\log_3 x^5 = \boxed{5 \cdot \log_3 x}$ or $\boxed{5\log_3 x}$

(8) $\log_4 \sqrt{x} = \log_4 x^{1/2}$

remember $\sqrt{x} = x^{1/2}$

$= \boxed{\frac{1}{2}\log_4 x}$

$\log_5\left(\frac{1}{\sqrt[3]{x}}\right)$

(9) $\log_7(a^b) = \boxed{b \cdot \log_7 a}$

Write each log as the sum and/or difference of multiples of logs.

(10) $\log_3\left(\frac{35}{44}\right) = \log_3\left(\frac{5 \cdot 7}{2^2 \cdot 11}\right)$

find prime factors

$= \log_3 5 + \log_3 7 - \log_3 2^2 - \log_3 11$

$= \boxed{\log_3 5 + \log_3 7 - 2\log_3 2 - \log_3 11}$

factors in numerator are added; factors in denom are subtracted

11) $\log_2 \dfrac{x^5}{(y+1)^2} = \log_2 x^5 - \log_2 (y+1)^2$

$= \boxed{5\log_2 x - 2\log_2(y+1)}$

Note: parentheses are required.
$\log_2 y+1$ means $\log_2(y) + 1$
or $\log_2(y) + \log_2(2)$

12) $\log_3\left(\dfrac{x+5}{x}\right)^2$

option 1: outside exponent first

$= 2\log_3\left(\dfrac{x+5}{x}\right)$

$= 2\left[\log_3(x+5) - \log_3(x)\right]$

$= \boxed{2\log_3(x+5) - 2\log_3(x)}$

option 2: use exponent properties first

$= \log_3 \dfrac{(x+5)^2}{x^2}$

$= \log_3(x+5)^2 - \log_3 x^2$

$= \boxed{2\log_3(x+5) - 2\log_3(x)}$

parentheses required

Given $\log_b 2 = 0.43$

and $\log_b 3 = 0.68$,

Use properties of logs to evaluate the following:

← Notice: base b not given. Don't know any logs except $\log_b 2$ and $\log_b 3$.

(13) $\log_b 6$

step 1: Write argument in prime factors and powers

$6 = 2 \cdot 3$

step 2: Rewrite log using prime factors

$= \log_b 2 \cdot 3$

step 3: Write as sum/difference of multiples of logs.

$= \log_b 2 + \log_b 3$

step 4: Substitute the given values

$= \log_b 2 + \log_b 3$

$\downarrow \qquad \downarrow$

$= 0.43 \quad 0.68$

step 5: Do arithmetic (must be easy — if you get a mile-long decimal, you made a mistake!)

$= \boxed{1.11}$

(14) $\log_b 9 = \log_b 3\cdot 3$ or $\log_b 3^2$

$= \log_b 3 + \log_b 3$

$= .68 + .68$

$= \boxed{1.36}$

$= 2\log_b 3$

$= 2(0.68)$

$= \boxed{1.36}$

(15) $\log_b \sqrt{2}$

$= \log_b 2^{1/2}$

$= \frac{1}{2}\log_b 2$

$= \frac{1}{2}(.43)$

$= \boxed{0.215}$

same instructions

(16) $\log_b 36$

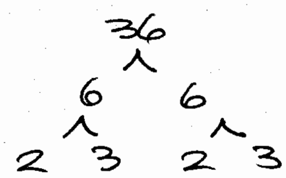
$= \log_b(2^2 \cdot 3^2)$

$= \log_b 2^2 + \log_b 3^2$

$= 2\log_b 2 + 2\log_b 3$

$= 2(0.43) + 2(0.68)$

$= \boxed{2.22}$



$36 = 2^2 \cdot 3^2$

(17) $\log_b 72$

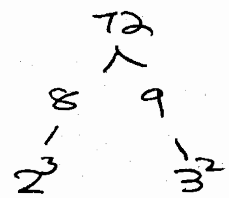
$= \log_b(2^3 \cdot 3^2)$

$= \log_b 2^3 + \log_b 3^2$

$= 3\log_b 2 + 2\log_b 3$

$= 3(0.43) + 2(0.68)$

$= \boxed{2.65}$



(18) $\log_b \frac{4}{9}$

$= \log_b \frac{2^2}{3^2}$

$= \log_b 2^2 - \log_b 3^2$

$= 2\log_b 2 - 2\log_b 3$

$= 2(.43) - 2(.68)$

$= \boxed{-0.5}$

same instructions, continued.

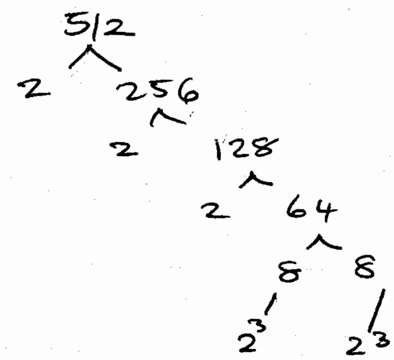
(19) $\log_b 512$

$= \log_b 2^9$

$= 9 \log_b 2$

$= 9(.43)$

$= \boxed{3.87}$



Challenge Problem

(20) $\log_b \sqrt[3]{1536}$

$= \log_b (1536)^{1/3}$

$= \frac{1}{3} \log_b 1536$

$= \frac{1}{3} \log_b (2^9 \cdot 3)$

$= \frac{1}{3} \log_b 2^9 + \frac{1}{3} \log_b 3$

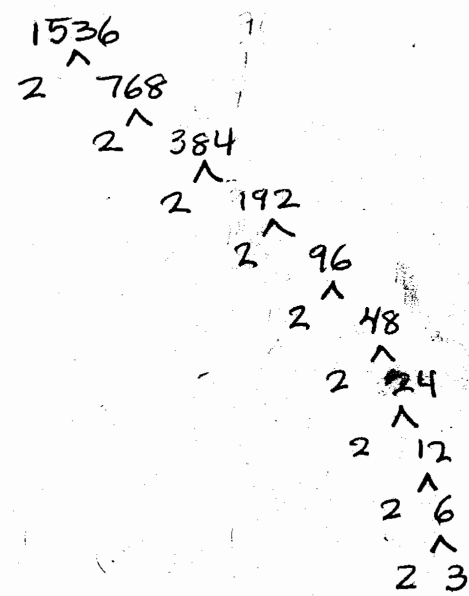
$= 9 \cdot \frac{1}{3} \cdot \log_b 2 + \frac{1}{3} \log_b 3$

$= 3 \log_b 2 + \frac{1}{3} \log_b 3$

$= 3(.43) + \frac{1}{3}(.68)$

$= \boxed{1.51\overline{6}}$ ← DO NOT ROUND!

$= \boxed{\frac{91}{60}}$



Write as a single log

(20) $\log_6 18 + 3\log_6 2 - \log_6 9$

$= \log_6 18 + \log_6 2^3 - \log_6 9$

$= \log_6 18 + \log_6 8 - \log_6 9$

$= \log_6\left(\frac{18 \cdot 8}{9}\right)$

$= \boxed{\log_6 16}$

(21) $\log_9 4x - \log_9 (x-3) + \log_9(x^3+1)$

$= \boxed{\log_9 \frac{4x\,(x^3+1)}{x-3}}$

Write as sum or difference or multiple logs

(22) $\log_6 \frac{x^2}{x+3}$

$= \log x^2 - \log_6(x+3)$

$= \boxed{2\log x - \log_6(x+3)}$

(23) $\log_5 x^3(x+1)$

$= \log_5 x^3 + \log_5(x+1)$

$= \boxed{3\log_5 x + \log_5(x+1)}$

(24) $\log_2 \frac{x^3}{\sqrt{y}}$

$= \log_2 x^3 - \log_2 \sqrt{y}$

$= \boxed{3\log_2 x - \frac{1}{2}\log_2 y}$

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Approximate $\log_2 3 = x$ to 4 decimal places.

[Technically $x = \log_2 3$ is "Solved". That's the exact answer.]

step 1: Write equivalent exponential equation

$$2^x = 3$$

step 2: Take common logs of both sides of equation.

$$\log 2^x = \log 3$$

step 3: Use property of logs on LHS

$$x \cdot \log 2 = \log 3$$

$\boxed{\text{NOTE}}$: log 2 is a number — a constant!
So is log 3! So we can isolate x by dividing both sides by that number

$$\frac{x \cdot \log 2}{\log 2} = \frac{\log 3}{\log 2}$$

CAUTION $\frac{\log 3}{\log 2} \neq \log \frac{3}{2}$

Since $\log \frac{3}{2} = \log 3 - \log 2$

$$\log_2 3 = x = \frac{\log 3}{\log 2} \quad \text{exact}$$

1.58496

$\approx \boxed{1.5850}$ approx

We used common logs because they're on the GC. But we could have used natural logs. Does the result change?

$$\log_2 3 = x \qquad \text{(It's already solved for } x\text{!)}$$

$$2^x = 3$$

$$\ln 2^x = \ln 3$$

$$x \ln 2 = \ln 3$$

$$\underset{\text{exact}}{\boxed{x = \log_2 3}} = \boxed{x = \frac{\ln 3}{\ln 2}} \text{ exact} \qquad 1.58496 = \boxed{1.5850} \text{ approx}$$

Solve ÷ logs (2 logs)

VS.

log prop ÷ argu- ments (1 log)

The exact expression is the same, but base e. The approximate result is identical.

CAUTION

log properties are different from change of base

$$\log_b a - \log_b c = \log_b\left(\frac{a}{c}\right)$$

$$\log 3 - \log 2 = \log\left(\frac{3}{2}\right)$$

$$\log\left(\frac{3}{2}\right) \approx .1761$$

If we'd used base 7 we'd get:

$$\log_2 3 = \frac{\log_7 3}{\log_7 2}$$

or any other base of logarithm.

Change of Base Formula

$$\log_b x = \frac{\log x}{\log b} \qquad \text{to change base } b \text{ to base } 10$$

$$\log_b x = \frac{\ln x}{\ln b} \qquad \text{to change base } b \text{ to base } e$$

$$\log_b x = \frac{\log_c x}{\log_c b} \qquad \text{to change base } b \text{ to base } c$$

Q: What is the difference, if any, between these?

a) $\frac{\log 3}{\log 2}$ b) $\log\left(\frac{3}{2}\right)$ c) $\frac{\log 3}{2}$

A: They are all different from each other!

a) $\frac{\log 3}{\log 2} = \log_2 3$ change of base formula

$\approx \underline{\underline{1.58496}}$

b) $\log\left(\frac{3}{2}\right) = \log(1.5)$ or $\log(3) - \log(2)$ log property $\log_b\left(\frac{a}{c}\right) = \log_b a - \log_b c$

$\approx \underline{\underline{0.17609}}$

c) $\frac{\log 3}{2} = \frac{\log(3)}{2} = \frac{1}{2}\log(3) = \log 3^{1/2} = \log\sqrt{3}$

arithmetic coefficient log property $k\log_b a = \log_b a^k$ ½ exponent means square root.

$\approx \underline{\underline{0.23856}}$

(25) Find approximate values. Round to nearest ten-thousandth.

a) $\log_5 7 = \dfrac{\log 7}{\log 5} \approx \boxed{1.2091}$

b) $\log_2 1 = \boxed{0}$ (log property!)

c) $\log_2 10 = \dfrac{\log 10}{\log 2} = \boxed{3.3219}$ not the same as $\log_{10} 2$!

(26)

Use GC to look at graph of $y = \log_2 x$.

$Y_1 = \log(x)/\log(2)$

– OR –

$Y_1 = \ln(x)/\ln(2)$.

(27)

Calculate $\log \sqrt[3]{10}$

a) without GC.

$\log \sqrt[3]{10} = \log_{10} 10^{1/3} = \boxed{\frac{1}{3}}$ inverse property $\log_b b^x = x$

b) with GC using cube root $\log(\sqrt[3]{10})$

$\boxed{\text{LOG}}$ $\boxed{\text{MATH}}$ 4. 10)) MATH 1.

$\uparrow$ $\sqrt[3]{\ }$ $\qquad$ $\uparrow$ >frac

$\log(\sqrt[3]{\ }(10))$ >frac

c) with GC using $\frac{1}{3}$ power $\log(10^{1/3})$

$\log(10^\wedge(1/3))$ >frac $= \boxed{\frac{1}{3}}$

CAUTION: NOT $\log(10)^\wedge(1/3)$ $\Rightarrow$ order of operations nested parentheses inside out